A Decomposition of Signed Graphs With Two Eigenvalues

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Abstract. In this study we consider connected signed graphs with 2 eigenvalues that admit a vertex set partition such that the induced signed graphs also have 2 eigenvalues, each. We derive some spectral characterizations, along with examples supported by additional theoretical results. We also prove an inequality that is a fundamental ingredient for the resolution of the Sensitivity Conjecture.

1. Introduction

Given a graph $G = (V(G), E(G))$, let $\sigma \colon E(G) \longrightarrow \{-1, 1\}$. Then $\dot{G} = (G, \sigma)$ is a *signed graph* derived from its *underlying graph* G. In other words, each edge e of G is accompanied by the sign $\sigma(e) \in \{-1, 1\}$. The (multiplicative) group $\{-1, 1\}$ can also be written as $\{-, +\}$. We also denote $|V(\dot{G})|$ by n. The set of edges of a signed graph is composed of the subset of positive edges and the subset of negative edges. Every graph is interpreted as a signed graph with all the edges being positive.

The $n \times n$ adjacency matrix $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard $(0, 1)$-adjacency matrix of G by reversing the sign of all 1s which correspond to negative edges. The eigenvalues of $A_{\dot{G}}$ are real and, together with their multiplicity, form the spectrum of $\dot{G}$.

A characterization of signed graphs with few (here and following, distinct) eigenvalues is listed as an open problem in [1]. In particular, signed graphs with 2 eigenvalues are considered before in [5, 11, 16]. For a similar study on graphs for which some universal matrix – a generalization of several matrices associated with graphs – has just 2 eigenvalues, we refer the reader to [9].

In this paper our focus is on connected signed graphs $\dot{G}$ with 2 eigenvalues that admit a vertex set partition such that the corresponding subgraphs – induced by the partition and denoted by $\dot{H}_1$ and $\dot{H}_2$ – have 2 eigenvalues, each. We briefly say that $\dot{G}$ is decomposed into $\dot{H}_1$ and $\dot{H}_2$. A study concerning decompositions of strongly regular unsigned graphs can be found in [8]. We give some spectral relations between $\dot{G}$, $\dot{H}_1$ and $\dot{H}_2$ from which one can conclude that, in general, spectra of two smaller signed graphs are not fully determined by the spectrum of $\dot{G}$. We proceed with selected (in our opinion, very illustrative) examples giving a general insight into the nature of signed graphs. There, we prove an inequality which yields the resolution of a conjecture posed in 1992 known as the Sensitivity Conjecture.

It occurs that our decompositions are closely related to star complements of signed graphs. The concept of star complements is transferred from the domain of graphs, along with some modifications. For the sake of completeness and to introduce necessary terminology and notation, we briefly describe the concept of star complements of signed graphs. All the basic results and some inconsistencies with the same concept in particular case of graphs are collected.

In Section 2 we list certain terminology and notation. Section 3 is devoted to star complements of signed graphs. Our main contribution, including the mentioned spectral relations and examples, is presented in Sections 4 and 5.

2. Preliminaries

We use the standard terminology and notation transferred from the domain of graphs – see [4]; for example, we say that a signed graph is connected or regular if the same holds for its underlying graph. If the vertices i and j are adjacent, we write $i \sim j$; in particular, the existence of a positive (resp. negative) edge between these vertices is designated by $i \overset{+}{\sim} j$ (resp. $i \overset{-}{\sim} j$).

For S subset of $V(\dot{G})$, let $\dot{G}^S$ be the signed graph obtained from $\dot{G}$ by reversing the sign of each edge between a vertex in S and a vertex in $V(\dot{G}) \setminus S$. The signed graph $\dot{G}^S$ is said to be *switching equivalent* to $\dot{G}$. Switching equivalent graphs share the same spectrum. We say that signed graphs $\dot{G}$ and $\dot{H}$ are *switching isomorphic* if one of them is isomorphic to the signed graph that is switching equivalent to the other; in that case, we write $\dot{G} \cong \dot{H}$. In the spectral context, one can frequently meet an approach which does not make any distinction between switching isomorphic signed graphs.

A cycle in a signed graph is said to be *positive* if the number of negative edges contained is not odd. Orherwise, it is said to be *negative*.

The *negation* $-\dot{G}$ of a signed graph $\dot{G}$ is obtained by reversing the sign of every edge of $\dot{G}$. The eigenvalues of $-\dot{G}$ are obtained by reversing the sign of the eigenvalues of $\dot{G}$. A *signed doubled graph* $\ddot{G}$ is obtained by doubling every edge of a graph G (i.e., a signed graph with all the edges being positive) with a negative edge; in fact, it is a signed multigraph.

We use $m_{\dot{G}}(\lambda)$ to denote the multiplicity of the eigenvalue λ in the spectrum of $\dot{G}$. ($\dot{G}$ is written in the subscript because, in what follows, we deal with different signed graphs sharing the same eigenvalue(s).) If $\dot{G}$ has exactly 2 eigenvalues, λ and μ, it follows that

$$m_{\dot{G}}(\lambda) = \frac{\mu n}{\mu - \lambda} \text{ and } m_{\dot{G}}(\mu) = \frac{\lambda n}{\lambda - \mu}. \tag{1}$$

In this paper we use the concept of signed line graphs that can be found in [2, 16, 17] and some other references. Introduce the vertex-edge orientation $\eta: V(\dot{G}) \times E(\dot{G}) \longrightarrow \{-1, 0, 1\}$ formed by obeying the following rules: (1) $\eta(i, jk) = 0$ if $i \notin \{j, k\}$, (2) $\eta(i, ij) = 1$ or $\eta(i, ij) = -1$ and (3) $\eta(i, ij)\eta(j, ij) = -\sigma(ij)$. The vertex-edge *incidence matrix* B_η is the matrix whose rows and columns are indexed by $V(\dot{G})$ and $E(\dot{G})$ respectively, such that its (i, e)-entry is equal to $\eta(i, e)$. Then, even if multiple edges exist in $\dot{G}$, we have

$$B_\eta^\intercal B_\eta = 2I + A_{L(\dot{G})}, \tag{2}$$

where $L(\dot{G})$ is taken to be a *signed line graph* of $\dot{G}$. A signed line graph defined in this way depends on orientation, but it is easy to show that all of them are switching equivalent. Since the matrix on the left-hand side of (2) is positive semidefinite, the least eigenvalue of a signed line graph is greater than or equal to -2. This definition of a signed line graph is tailored for the spectral theory and differs in sign from the one introduced by Zaslavsky [17]. To avoid possible confusion, we suggest that in a wider context $L(\dot{G})$ (defined above) could be called a *spectral line graph*.

3. Star complements in signed graphs

Given a signed graph $\dot{G}$ with n vertices, let λ be its eigenvalue and P a matrix representing the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\lambda)$ with respect to the canonical basis $\{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_n}\}$. There is a set

$S \subseteq V(\dot{G}) = \{1, 2, \ldots, n\}$, such that the vectors $P\mathbf{e_i}$ $(i \in S)$ make a basis of $\mathcal{E}(\lambda)$. Such a set S is called a *star set* for λ, while the signed graph induced by the set $\overline{S} = V(\dot{G}) \setminus S$ is called a *star complement* for λ. It follows that λ is not an eigenvalue of the star complement and $|S| = m_{\dot{G}}(\lambda)$. We denote $k = n - m_{\dot{G}}(\lambda)$.

Since the adjacency matrix $A_{\dot{G}}$ and the matrix P commute, we have

$$\lambda P\mathbf{e_i} = PA_{\dot{G}}\mathbf{e_i} = \sum_{j \sim i} \sigma(ij)P\mathbf{e_j}. \tag{3}$$

The three forthcoming results are, with slight modifications, transferred from the filed of graphs. The proofs are almost unchanged; to demonstrate this, we include the proof of the first statement.

Theorem 3.1. (cf. [4, Proposition 5.1.4]) *Let S be a star set for λ in $\dot{G}$.*

(i) *If $\lambda \neq 0$, then every vertex of S is adjacent to at least one vertex of $\overline{S}$.*

(ii) *If $\lambda \notin \{-1, 0, 1\}$, then each two vertices of S have distinct neighbourhoods in $\overline{S}$.*

Proof. (i): From (3) we have that, for $\lambda \neq 0$ and $u \in S$, the vectors $P\mathbf{e_u}, P\mathbf{e_j}$ $(j \sim u)$ are linearly dependent. Since the vectors $P\mathbf{e_i}$ $(i \in S)$ are linearly independent (see definition of S), it follows that u is adjacent to a vertex in $\overline{S}$.

(ii): If $u, v \in S$ are the vertices sharing the same neighbourhood in $\overline{S}$, then by (3) we have

$$\lambda(P\mathbf{e_u} - P\mathbf{e_v}) - \sum_{i \in S, \, i \sim u} \sigma(ui)P\mathbf{e_i} + \sum_{i \in S, \, i \sim v} \sigma(vi)P\mathbf{e_i} = \mathbf{0}.$$

Linear independence of the vectors on the left hand side yields

$$\begin{cases} \lambda = 0 & \text{and} \quad u \nsim v, \\ \lambda = -1 & \text{and} \quad u \overset{+}{\sim} v \text{ or} \\ \lambda = 1 & \text{and} \quad u \overset{-}{\sim} v, \end{cases}$$

completing (ii). $\square$

Continue with the following theorem.

Theorem 3.2. [4, Theorem 5.1.7] *Given a signed graph $\dot{G}$ with the adjacency matrix*

$$\begin{pmatrix} A_S & B^{\mathsf{T}} \\ B & A_{\dot{H}} \end{pmatrix},$$

where A_S is the $m_{\dot{G}}(\lambda) \times m_{\dot{G}}(\lambda)$ adjacency matrix of the subgraph induced by a vertex set S, while $\dot{H}$ is the subgraph induced by $V(\dot{G}) \setminus S$. Then S is a star set for λ if and only if λ is not an eigenvalue of $\dot{H}$ and

$$\lambda I - A_S = B^{\mathsf{T}}(\lambda I - A_{\dot{H}})^{-1}B.$$

This result is called the Reconstruction Theorem. The proof can be derived by following the corresponding reference.

If $\dot{H}$ is a star complement for an eigenvalue λ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, then we define the following bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathsf{T}}(\lambda I - A_{\dot{H}})^{-1}\mathbf{y}.$$

Here is a direct consequence.

Corollary 3.3. (cf. [4, Corollary 5.1.9]) *If λ is not an eigenvalue of a signed graph $\dot{H}$ (with k vertices), then there is a signed graph $\dot{G}$ with $\dot{H}$ as a star complement for λ if and only if*

$$\langle \mathbf{b_i}, \mathbf{b_i} \rangle = \lambda \quad and \quad \langle \mathbf{b_i}, \mathbf{b_j} \rangle \in \{-1, 0, 1\},$$

for all distinct $i, j \in S = V(\dot{G}) \setminus V(\dot{H})$, where $\mathbf{b_i}$ and $\mathbf{b_j}$ determine neighbourhoods of i and j in $\dot{H}$, respectively.

Clearly, the vectors $\mathbf{b_i}$ $(1 \leq i \leq n - k)$ form the submatrix B of the reconstruction theorem. Also, if $\langle \mathbf{b_i}, \mathbf{b_j} \rangle = 0$ (resp. $\langle \mathbf{b_i}, \mathbf{b_j} \rangle = -1$, $\langle \mathbf{b_i}, \mathbf{b_j} \rangle = 1$), then $i \nsim j$ (resp. $i \overset{+}{\sim} j$, $i \overset{-}{\sim} j$).

The matrix $(\lambda I - A_{\dot{H}})^{-1}$ can be computed by [4, Proposition 5.1.11]; the method remains valid for signed graphs.

Remark 3.4. *It is worth mentioning that the quadratic upper bound for n expressed in terms of $|S|$, that can be found in [4, Proposition 5.1.10], does not hold for signed graphs. Certain counterexamples are given in Subsection 5.3. In fact, there is a sharp cubic upper bound proved in [13].*

Finally, an observation related to signed graphs.

Corollary 3.5. *A signed graph $\dot{G}$ contains a star complement $\dot{H}$ for λ if and only if $-\dot{G}$ contains a star complement $-\dot{H}$ for $-\lambda$.*

Proof. The equivalence follows since $m_{\dot{G}}(\lambda) = m_{-\dot{G}}(-\lambda)$. $\square$

4. Spectral relations

By considering the minimal polynomial of the adjacency matrix, we easily conclude that a signed graph with 2 eigenvalues, say λ and μ, is regular with vertex degree $-\lambda\mu$. We proceed with the following result.

Theorem 4.1. *Let a signed graph $\dot{G}$ with 2 eigenvalues, λ and μ, be decomposed into signed graphs $\dot{H}_1$ and $\dot{H}_2$. If ν is an eigenvalue of $\dot{H}_1$ and $\nu \notin \{\lambda, \mu\}$, then $\lambda + \mu - \nu$ is an eigenvalue of $\dot{H}_2$. In addition, $m_{\dot{H}_1}(\nu) = m_{\dot{H}_2}(\lambda + \mu - \nu)$.*

Proof. Expressing $A_{\dot{G}}$ in the form

$$A_{\dot{G}} = \begin{pmatrix} A_{\dot{H}_1} & B^\top \\ B & A_{\dot{H}_2} \end{pmatrix}, \tag{4}$$

and using $A_{\dot{G}}^2 = (\lambda + \mu)A_{\dot{G}} - \lambda\mu I$, we get $A_{\dot{H}_2}B = -BA_{\dot{H}_1} + (\lambda + \mu)B$. Now, if ν is an eigenvalue of $\dot{H}_1$ and $\mathbf{x}$ is an associated eigenvector, then

$$\begin{aligned} A_{\dot{H}_2}B\mathbf{x} &= -BA_{\dot{H}_1}\mathbf{x} + (\lambda + \mu)B\mathbf{x} \\ &= -\nu B\mathbf{x} + (\lambda + \mu)B\mathbf{x} \\ &= (\lambda + \mu - \nu)B\mathbf{x}, \end{aligned} \tag{5}$$

which means that $\lambda + \mu - \nu$ is an eigenvalue of $A_{\dot{H}_2}$, unless $B\mathbf{x} = \mathbf{0}$. In the last case, the identity

$$A_{\dot{G}} \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} A_{\dot{H}_1}\mathbf{x} \\ \mathbf{0} \end{pmatrix} = \nu \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix},$$

gives $\nu \in \{\lambda, \mu\}$.

Finally, if the eigenspace $\mathcal{E}_{\dot{H}_1}(\nu)$ of $\nu \notin \{\lambda, \mu\}$ is spanned by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, then, since the kernel of B trivially intersects the space $\mathcal{E}_{\dot{H}_1}(\nu)$, the vectors $B\mathbf{x}_1, B\mathbf{x}_2, \ldots, B\mathbf{x}_k$ are linearly independent, and so $\dim(\mathcal{E}_{\dot{H}_1}(\nu)) \leq \dim(\mathcal{E}_{\dot{H}_2}(\lambda + \mu - \nu))$. By interchanging $\dot{H}_1$ and $\dot{H}_2$, we arrive at the opposite inequality, which concludes the proof. $\square$

We proceed by a simple consequence.

Corollary 4.2. *Let a signed graph $\dot{G}$ with 2 eigenvalues be decomposed into signed graphs $\dot{H}_1$ and $\dot{H}_2$. There is an equal number of eigenvalues of $\dot{H}_1$ and $\dot{H}_2$ which do not appear in the spectrum of $\dot{G}$.*

Proof. The result follows by Theorem 4.1, since the existence of an eigenvalue of $\dot{H}_1$, but not of $\dot{G}$, implies the existence of an eigenvalue of $\dot{H}_2$, but not of $\dot{G}$, and vice versa. $\square$

Let further $\dot{H}_1$ and $\dot{H}_2$ also have 2 eigenvalues, each. Relations between theirs and the spectrum of $\dot{G}$ are given in our next result. Clearly, the case when $\dot{G}$ is connected is of particular interest.

Theorem 4.3. *Assume that a connected signed graph $\dot{G}$ with n vertices and eigenvalues λ and μ is decomposed into signed graphs $\dot{H}_1$ (with n_1 vertices and eigenvalues λ_1 and μ_1) and $\dot{H}_2$ (with n_2 vertices and eigenvalues λ_2 and μ_2).*

(i) *The case $\lambda_1, \mu_1 \in \{\lambda, \mu\}$ cannot occur.*

(ii) *If $\lambda_1, \mu_1 \notin \{\lambda, \mu\}$, then the spectrum of $\dot{G}$ is symmetric, $\dot{H}_2 \cong -\dot{H}_1$, and for $i, j \in V(\dot{H}_1)$ the columns of B (from (4)) satisfy*

$$\mathbf{b_i}^\top \mathbf{b_j} = \begin{cases} 0 & \text{if } i \not\sim j, \\ -\lambda_1 - \mu_1 & \text{if } i \overset{+}{\sim} j, \\ \lambda_1 + \mu_1 & \text{if } i \overset{-}{\sim} j. \end{cases}$$

(iii) *If $\lambda_1 = \lambda$ and $\mu_2 = \mu$, then $n_1\lambda_1 = -n_2\mu$, $\mu_1 + \lambda_2 = \lambda + \mu$, $m_{\dot{H}_1}(\mu_1) = m_{\dot{H}_2}(\lambda_2)$, $m_{\dot{G}}(\lambda) = n_1$ and $m_{\dot{G}}(\mu) = n_2$.*

(iv) *If $\lambda_1 = \lambda$ and $\lambda_2 = \lambda$, then $n_1(\lambda - \mu_2) = n_2(\lambda - \mu_1)$, $\mu_1 + \mu_2 = \lambda + \mu$, and $m_{\dot{H}_1}(\mu_1)$, $m_{\dot{H}_2}(\mu_2)$ and $m_{\dot{G}}(\mu)$ are equal.*

Proof. (i): Otherwise, $\dot{H}_1$ and $\dot{G}$ would have the same vertex degree, which – since $\dot{G}$ is connected – means that $\dot{H}_2$ is totally disconnected, so has exactly one eigenvalue.

(ii): By virtue of Corollary 4.2, $\dot{G}$ has $2n_1$ vertices. If its spectrum is non-symmetric, then the multiplicity of an eigenvalue would be greater than n_1, which is impossible since $\dot{G}$ contains an induced signed subgraph having n_1 vertices and avoiding that eigenvalue in the spectrum.

Since the spectrum of $\dot{G}$ is symmetric, from (4), we get

$$A_{\dot{H}_2}B = -BA_{\dot{H}_1}. \tag{6}$$

According to the assumption that $\dot{H}_2$ does not share any eigenvalue with $\dot{G}$, there are n_1 independent eigenvectors of the form $B\mathbf{x}$ (see (5)) associated with its eigenvalues. As these eigenvectors lie in the image of B, it follows that B is invertible. Then, by (6), we get $A_{\dot{H}_2} = -BA_{\dot{H}_1}B^{-1}$, giving $\dot{H}_2 \cong -\dot{H}_1$.

Observe that $\dot{H}_2$ is a star complement in $\dot{G}$, for both λ and $\mu = -\lambda$. By Corollary 3.3, for $i, j \in V(\dot{H}_1)$, the columns $\mathbf{b_i}$ and $\mathbf{b_j}$ of B satisfy $\langle \mathbf{b_i}, \mathbf{b_j} \rangle \in \{-1, 0, 1\}$, where the bilinear form is formed by $A_{\dot{H}_2}$ and each of $\lambda, -\lambda$. This gives

$$\mathbf{b_i}^\top (A_{\dot{H}_2} + (\pm\lambda - \lambda_2 - \mu_2)I)\mathbf{b_j} = p\phi_{\dot{H}_2}(\pm\lambda),$$

where $\phi_{\dot{H}_2}$ stands for the minimal polynomial of $\dot{H}_2$, while p is 0, -1 or 1 depending on whether $i \not\sim j$, $i \overset{+}{\sim} j$ or $i \overset{-}{\sim} j$. This yields $\lambda\mathbf{b_i}^\top\mathbf{b_j} = p\lambda(\lambda_2 + \mu_2)$, and since $\lambda \neq 0$ and the eigenvalues of $\dot{H}_1$ are obtained by reversing the sign of those of $\dot{H}_2$, (ii) is completed.

(iii), (iv): By Theorem 4.1, the eigenvalues of $\dot{H}_1$ and $\dot{H}_2$ which do not appear in the spectrum of $\dot{G}$ have the same multiplicity, and their sum is equal to $\lambda + \mu$.

Counting the number of edges between $\dot{H}_1$ and $\dot{H}_2$, we get $n_1(-\lambda\mu + \lambda_1\mu_1) = n_2(-\lambda\mu + \lambda_2\mu_2)$, which gives $n_1\lambda_1 = -n_2\mu$ for (iii), and $n_1(\lambda - \mu_2) = n_2(\lambda - \mu_1)$ for (iv).

To conclude (iii), it remains to compute the multiplicities of λ and μ in $\dot{G}$. First, by the eigenvalue interlacing, the multiplicity of each of them is not less than the multiplicity of the same eigenvalue in $\dot{H}_1$ and $\dot{H}_2$, respectively. Hence, we may write $m_{\dot{G}}(\lambda) = \frac{\mu_1 n_1}{\mu_1 - \lambda} + a$ and $m_{\dot{G}}(\mu) = \frac{\lambda_2 n_2}{\lambda_2 - \mu} + b$, for $a, b \geq 0$; if necessary, compare (1). Using the identities $\text{tr}(A_{\dot{G}}) = 0$ and $\text{tr}(A_{\dot{G}}^2) = -\lambda\mu(n_1 + n_2)$, we arrive at $a = b = \frac{\lambda n_1}{\lambda - \mu_1}$, giving $m_{\dot{G}}(\lambda) = n_1$, and so $m_{\dot{G}}(\mu) = n_2$.

For (iv), by setting $m_{\dot{G}}(\lambda) = \frac{\mu_1 n_1}{\mu_1 - \lambda} + \frac{\mu_2 n_2}{\mu_2 - \lambda} + a$ and using $\text{tr}(A_{\dot{G}}) = 0$, we get $a = \frac{\lambda(n_1 + n_2)}{\lambda - \mu}$. Using the equalities of (iv) we already proved, we get $m_{\dot{G}}(\lambda) = n_1 + n_2 - \frac{\lambda n_1}{\lambda - \mu_1}$, giving $m_{\dot{G}}(\mu) = \frac{\lambda n_1}{\lambda - \mu_1}$, i.e., $m_{\dot{G}}(\mu)$ is equal to the multiplicity of μ_1 in $\dot{H}_1$. This completes (iv) and the entire proof. $\square$

By Corollary 4.2, the previous theorem covers all the possibilities for the eigenvalues of $\dot{H}_1$ and $\dot{H}_2$.

5. Examples

At this point, the next natural step could be creating a list of parameter sets for a limited number of vertices. In fact, they can easily be obtained on the basis of the results of the previous section. Here we skip their presentation and restrict ourselves to the certain examples supported by additional theoretical results.

5.1. For Theorem 4.3(ii)

Lemma 5.1. *Assume that $\dot{H}$ is a signed graph with 2 eigenvalues and that $\dot{G}$ is determined by*

$$A_{\dot{G}} = \begin{pmatrix} A_{\dot{H}} & B \\ B & -A_{\dot{H}} \end{pmatrix},$$

where B commutes with $A_{\dot{H}}$ and satisfies the condition of Theorem 4.3(ii), while $|B|$ has a constant row sum. Then $\dot{G}$ has exactly 2 (symmetric) eigenvalues of the form $\pm\sqrt{a + b}$, where a and b denote the row sums of $|A_{\dot{H}}|$ and $|B|$, respectively.

Proof. Since the blocks of $xI - A_{\dot{G}}$ commute, we have

$$\left| xI - A_{\dot{G}} \right| = \left| (xI - A_{\dot{H}})(xI + A_{\dot{H}}) - B^2 \right| = \left| x^2 I - \left(A_{\dot{H}}^2 + B^2 \right) \right|.$$

Due to the remaining assumptions (on B), the last determinant is equal to $|(x^2 - (a + b))I|$, where a and b denote the row sums of $|A_{\dot{H}}|$ and $|B|$, respectively. (Recall that, since $\dot{H}$ has 2 eigenvalues, it is regular, and so $|A_{\dot{H}}|$ has a constant row sum). Therefore, $\dot{G}$ has 2 eigenvalues (of the form $\pm\sqrt{a + b}$). □

The following example arises naturally.

Example 5.2. If $\dot{H}$ is a signed graph with 2 symmetric eigenvalues and vertex degree r, by taking $B = \pm A_{\dot{H}}$ or $B = \pm I$ in Lemma 5.1, we obtain a signed graph $\dot{G}$ with 2 eigenvalues: $\pm\sqrt{2r}$ in the first case and $\pm\sqrt{r+1}$ in the second. Indeed, B obviously satisfies all the assumptions of the corresponding lemma, while the eigenvalues are computed as in the same result.

In this way we obtain infinite families of decompositions $(\dot{G}^i, \dot{H}^i, -\dot{H}^i)$, $i \geq 0$. Namely, it is sufficient to take $\dot{H}^0$ to be any signed graph with 2 symmetric eigenvalues, obtain $\dot{G}^0$ by choosing B as above and set $\dot{H}^{i+1} = \dot{G}^i$.

Recall that the r-dimensional cube Q_r is the r-regular graph of order 2^r with the vertex set $\{0, 1\}^r$ (all possible binary r-tuples) in which two vertices are adjacent if they differ in exactly one coordinate. Accordingly, an r-dimensional signed cube is a signed graph whose underlying graph is Q_r. In [5], the authors studied these signed graphs and proved that, for $2 \leq r \leq 4$, the r-dimensional signed cube with negative quadrangles has 2 eigenvalues. We generalize this result.

Theorem 5.3. *For every $r \geq 2$, up to the switching isomorphism, there exists a unique r-dimensional signed cube $\dot{Q}_r$ with negative quadrangles. Its eigenvalues are $\pm\sqrt{r}$.*

Proof. Obviously, $\dot{Q}_2$ is isomorphic to a negative quadrangle, all negative quadrangles are switching isomorphic and their common eigenvalues are $\sqrt{2}$ and $-\sqrt{2}$.

Assume that our statement holds for $\dot{Q}_{r-1}$. If its colour classes are denoted by U_1 and U_2, then it is a matter of routine to verify that $\dot{Q}_r$ is obtained by taking the two copies of $\dot{Q}_{r-1}$ and inserting a positive (resp. negative) edge between the corresponding copies of a vertex u if $u \in U_1$ (resp. $u \in U_2$). Its uniqueness – up to the switching isomorphism – follows by the way of construction.

It remains to consider the eigenvalues. The adjacency matrix of $\dot{Q}_r$ is the first matrix from below, where the 2×2 top-left block and its diagonal counterpart both stand for the adjacency matrix of $\dot{Q}_{r-1}$.

$$\begin{pmatrix} O & N^\top & I & O \\ N & O & O & -I \\ I & O & O & N^\top \\ O & -I & N & O \end{pmatrix} \quad \begin{pmatrix} O & N^\top & I & O \\ N & O & O & I \\ I & O & O & -N^\top \\ O & I & -N & O \end{pmatrix}$$

The second matrix is obtained by applying the switching with respect to the vertices of one colour class of the second copy of $\dot{Q}_{r-1}$. The result follows by Lemma 5.1 applied to this matrix. □

Here is a simple, but significant, corollary.

Corollary 5.4. *If H is a subgraph of Q_r induced by any set of $2^{r-1} + 1$ vertices, then the maximum vertex degree of H, $\Delta(H)$, satisfies $\Delta(H) \geq \sqrt{r}$.*

Proof. If $\dot{Q}_r$ is the r-dimensional signed cube with negative quadrangles and $\dot{H}$ is its subgraph sharing the vertex set with H, then by Theorem 5.3 and the interlacing argument, the largest eigenvalue of $\dot{H}$ is $\geq \sqrt{r}$, which means that the largest eigenvalue of its underlying graph H is also $\geq \sqrt{r}$ (as the largest eigenvalue of a signed graph never exceeds the largest eigenvalue of its underlying graph [14]), and consequently we get $\Delta(H) \geq \sqrt{r}$. □

Remark 5.5. *The inequality of Corollary 5.4 is a fundamental ingredient for the resolution of the Sensitivity Conjecture posed by Nisan and Szegedy in 1992 [12]. The resolution can be found in Huang [10].*

There are examples even if the spectrum of $\dot{H}$ is non-symmetric. By computer search, we found that a signed graph with 20 vertices and eigenvalues $\pm\sqrt{10}$ can be decomposed into $L(K_5)$ (with eigenvalues 3 and -2) and its negation.

5.2. For Theorem 4.3(iii)

Here is just one example. (In the exponential notation, the exponent denotes the multiplicity of the corresponding eigenvalue.)

Example 5.6. For $n \geq 3$, the signed line graph $L(K_{n+1})$ (with spectrum $[(n-1)^n, (-2)^{\binom{n}{2}}]$) can be decomposed into $L(K_n)$ (with spectrum $[(n-2)^{n-1}, (-2)^{\binom{n-1}{2}}]$) and K_n (with spectrum $[n-1, (-1)^{n-1}]$).

Indeed, the edges incident with a fixed vertex of K_{n+1} induce K_n in $L(K_{n+1})$, while the remaining edges induce $L(K_n)$ in the same signed line graph. Both K_n and $L(K_n)$ are star complements in $L(K_{n+1})$ (for distinct eigenvalues).

5.3. For Theorem 4.3(iv)

In [16], we proved that a connected signed graph $L(\ddot{G})$ has exactly 2 eigenvalues, whenever G is an r-regular graph with n ($n \geq 3$) vertices. Moreover, we have $B_\eta B_\eta^\top = 2rI$ (B_η being defined in Section 2), which together with (2) gives the characteristic polynomial

$$\Phi_{L(\ddot{G})} = (x+2)^{n(r-1)} (x - 2(r-1))^n. \tag{7}$$

Using this, we arrive at a very rich family of decompositions.

Example 5.7. Take any regular graph G which contains 2 connected regular spanning subgraphs H_1 and H_2, both with vertex degree at least 2 and such that every edge of G belongs to exactly one of them – there are plenty of possibilities. Then, $L(\ddot{G})$ is decomposed into $L(\ddot{H}_1)$ and $L(\ddot{H}_2)$, and each of them has 2 eigenvalues which can be computed by (7).

If we allow H_2 to be induced by a perfect matching of G, then we arrive at a decomposition into $L(\ddot{H}_1)$ (with 2 eigenvalues, unless G is an even cycle) and a totally disconnected signed graph $L(\ddot{H}_2)$ (with just one eigenvalue).

Remark 5.8. *Observing that the formula (7) depends only on n and r, we obtain an easy way to construct switching non-isomorphic signed graphs that share the same spectrum. Only what we need is to select a pair of non-isomorphic regular graphs with equal number of vertices and equal vertex degree. If these graphs are denoted by G and H, then $L(\ddot{G})$ and $L(\ddot{H})$ are the desired signed graphs. They are non-isomorphic because their underlying graphs are non-isomorphic.*

We continue with decompositions including connected signed graphs which are not signed line graphs, but their eigenvalues are not less than −2 (they are, so-called, exceptional signed graphs). It is known, say from [3], that every exceptional signed graph has a representation in the root system E_8; we believe that the reader is familiar with this and similar systems.

Any set of (the 120) positive roots of E_8 represents a signed graph, say $\dot{M}_8$, with spectrum $[28^8, (−2)^{112}]$. (In other words, we have $A_{\dot{M}_8} = N^\intercal N − 2I$, where N is obtained by arranging all the positive roots as its columns.) In addition, signed graphs obtained for different choices of positive roots are switching isomorphic. Similarly, any set of (the 63) positive roots of E_7 represents a signed graph, $\dot{M}_7$, with spectrum $[16^8, (−2)^{56}]$.

Example 5.9. Signed graphs with eigenvalues ± 2 are fully determined in [16] (the main part of this result has been reported earlier in [5]; for a more general approach, the reader can consult [11]). There is an infinite family of such signed graphs, and it occurs that exactly 2 of them are exceptional: the first has 14 vertices and spectrum $[2^7, (−2)^7]$, and the second (the 4-dimensional cube with negative quadrangles) has 16 vertices and spectrum $[2^8, (−2)^8]$.

In what follows, we refer to the list of positive roots of E_8 expressed on the basis of fundamental roots; for example, given in [6].

The exceptional signed graph with 16 vertices is represented by the roots $1, 2, 3, 4, 6, 15, 18, 21, 44, 48, 49, 86, 87, 97, 119$ and 120 of [6, Appendix B]. By taking the remaining positive roots, we arrive at the signed graph with spectrum $[24^8, (−2)^{96}]$. Clearly, this pair of signed graphs decomposes $\dot{M}_8$ (in the sense of Theorem 4.3(iv)).

Similarly, the exceptional signed graph with 14 vertices is represented by the roots $1, 2, 3, 4, 6, 7, 18, 21, 44, 48, 49, 80, 82$ and 97 of [6, Appendix B], and all these roots belong to some E_7. By taking the remaining positive roots of that E_7, we get the signed graph with spectrum $[12^7, (−2)^{42}]$ and complete a decomposition of $\dot{M}_7$.

The exceptional signed graph with 16 vertices is the 8-vertex extension of the exceptional star complement enumerated by 641 in [7, Table 3]. The other exceptional signed graph is the 7-vertex extension of each of the exceptional star complements enumerated by 1, 111, 112 and 113 in [7, Table 2].

Example 5.10. By (7), $L(\ddot{C}_8)$ shares the spectrum with the exceptional signed graph with 16 vertices mentioned in the previous example. To reveal its representation in E_8, it is convenient to express the roots as linear combinations of the canonical basis of $\mathbb{R}^8$; if necessary, see [4, 7, 15]. Then we easily conclude that $L(\ddot{C}_8)$ is represented by the roots $\mathbf{e_i} \pm \mathbf{e_{i+1}}$, where $1 \le i \le 8$ and the vector index larger than 8 is taken to be mod 8. Again, by taking the remaining positive roots, we get another decomposition of $\dot{M}_8$. The two larger signed graphs – obtained in this and the previous example, both with 104 vertices – share the same spectrum, but they are switching non-isomorphic.

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