

Research Article

Open Access

Special Issue Dedicated to Charles R. Johnson

Milica Anđelić, Tamara Koledin, and Zoran Stanić

A note on the eigenvalue free intervals of some classes of signed threshold graphs

<https://doi.org/10.1515/spma-2019-0014>

Received September 2, 2019; accepted November 21, 2019

Abstract: We consider a particular class of signed threshold graphs and their eigenvalues. If \dot{G} is such a threshold graph and $Q(\dot{G})$ is a quotient matrix that arises from the equitable partition of \dot{G} , then we use a sequence of elementary matrix operations to prove that the matrix $Q(\dot{G}) - xI$ ($x \in \mathbb{R}$) is row equivalent to a tridiagonal matrix whose determinant is, under certain conditions, of the constant sign. In this way we determine certain intervals in which \dot{G} has no eigenvalues.

Keywords: signed graph; threshold graph; tridiagonal matrix; eigenvalue interval

MSC: 05C50, 05C22

Dedicated to Charles R. Johnson on the occasion of his 70th birthday.

1 Introduction

A *signed graph* \dot{G} is a pair (G, σ) , where $G = (V, E)$ is an (unsigned) graph, and $\sigma: E \rightarrow \{+1, -1\}$ is the *signature* or the *sign function*. The edge set of a signed graph is composed of subsets of positive and negative edges. Throughout the paper we interpret a graph as a signed graph with all the edges being positive.

The *adjacency matrix* $A_{\dot{G}}$ of \dot{G} is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1's which correspond to negative edges. The *eigenvalues* of \dot{G} are identified to be the eigenvalues of $A_{\dot{G}}$.

A *threshold graph* (also known as a *nested split graph*) is a $\{2K_2, P_4, C_4\}$ -free graph, i.e., it does not contain any of listed graphs as an induced subgraph. To explain the structure of threshold graphs, we partition their set of vertices into the $2h$ cells $U_1, U_2, \dots, U_h, V_1, V_2, \dots, V_h$. Now, the subgraph induced by $\bigcup_{i=1}^h V_i$ (resp. $\bigcup_{i=1}^h U_i$) is complete (resp. totally disconnected), while all the vertices in U_i are adjacent to all the vertices in V_j precisely when $i \leq j$. The size of U_i is denoted by t_i and the size of V_j is denoted by s_j . A schematic representation of a threshold graph is illustrated in Fig 1. For more details on these graphs (including a wide branch of applications) we refer the reader to [10, 11].

Observe that a threshold graph is connected whenever $s_h \geq 1$. Since every disconnected threshold graph is composed of a connected one and a set of isolated vertices, in this paper we restrict ourselves only to connected ones. We also assume that all the cells are non-empty.

In what follows we consider the class of signed threshold graphs with the following signature:

- the edges between different cells are always positive;
- the edges within each cell V_i are either all positive or all negative.

We denote this class by \mathcal{T} .

Milica Anđelić: Department of Mathematics, Kuwait University, Safat 13060, Kuwait, E-mail: milica.andelic@ku.edu.kw

Tamara Koledin: Faculty of Electrical Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, Belgrade, Serbia, E-mail: tamara@etf.rs

Zoran Stanić: Faculty of Mathematics, University of Belgrade, Studentski trg 16, Belgrade, Serbia, E-mail: zstanic@math.rs

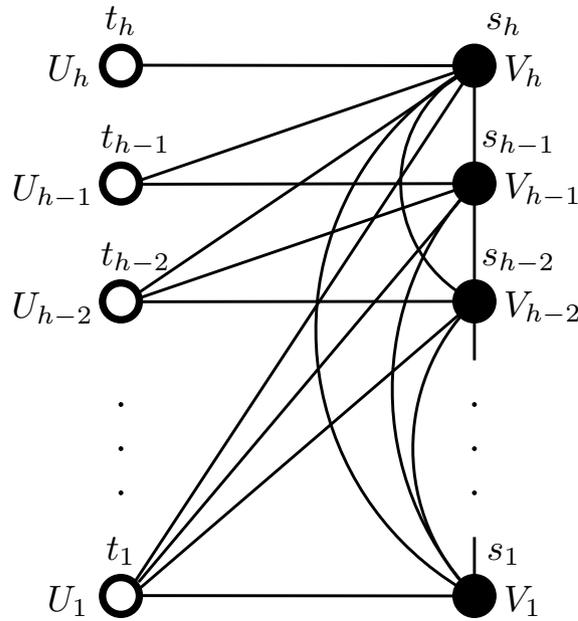


Figure 1: The structure of a threshold graph.

The problem of finding eigenvalue-free intervals has already been considered for threshold graphs with respect to the spectrum of the adjacency matrix in [1, 6, 7]. If we restrict ourselves to signed graphs, then there are no relevant references. This paper is prepared in the same spirit as [2], where the eigenvalue-free interval for threshold graphs (and chain graphs) with respect to distance spectrum were considered.

The paper is organized as follows. Two preliminary results are stated in the short Section 2. In Section 3 we consider the characteristic polynomial of the adjacency matrix of a signed threshold graphs defined as above. In Section 4 we deal with the interval $(-1, 0)$, and we also consider the multiplicity of the eigenvalues 0 and -1 . In Section 5 we show that in certain intervals the characteristic polynomials of our signed threshold graphs have constant sign, which proves that these intervals are eigenvalue free. Finally, in Section 6 we give some concluding remarks on multiplicity of the eigenvalues.

2 Background

Let M denote an $n \times n$ real symmetric matrix, and let its rows and columns be indexed by $S = \{1, 2, \dots, n\}$. Let further $S_1 \sqcup S_2 \sqcup \dots \sqcup S_k$ be a partition of S , which determines a blocking of M into the blocks $M_{i,j}$. If each block $M_{i,j}$ has a constant row sum, say $m_{i,j}$, the corresponding partition is called *equitable*, and the $k \times k$ matrix $Q = [m_{i,j}]$ is called the *quotient matrix* of M . The following result can be found in Section 2.3 of [3] (see also [4]).

Theorem 2.1. [3] *Let M be a real symmetric matrix with a quotient matrix Q . Then the characteristic polynomial of Q divides the characteristic polynomial of M .*

The next theorem gives sufficient conditions for a tridiagonal matrix (with positive diagonal entries) to have positive determinant.

Theorem 2.2. [9] *Let $M = [m_{i,j}]$ be an $n \times n$ real tridiagonal matrix with positive diagonal entries. If*

$$m_{i,i-1}m_{i-1,i} < \frac{1}{4}m_{i,i}m_{i-1,i-1} \frac{1}{\cos^2(\frac{\pi}{n+1})}, \text{ for } 2 \leq i \leq n,$$

then $\det(M) > 0$.

3 The characteristic polynomial of signed graphs in \mathcal{T}

Let u and v be the vertices belonging to the same cell of a signed threshold graph $\dot{G} \in \mathcal{T}$. Consider the vector \mathbf{x} defined in the following way: all its coordinates are zero except for those corresponding to u and v which are taken to be 1 and -1 . If \mathbf{w} is the row of $A_{\dot{G}}$ corresponding to some vertex $w \notin \{u, v\}$, then $\mathbf{w} \cdot \mathbf{x} = 0$ (where \cdot stands for the standard inner product). Also, if \mathbf{u} (resp. \mathbf{v}) is the row of $A_{\dot{G}}$ corresponding to the vertex u (resp. v), then $\mathbf{u} \cdot \mathbf{x} = \mathbf{v} \cdot \mathbf{x} = 0$, if $u, v \in U_i$, $\mathbf{u} \cdot \mathbf{x} = 1$, $\mathbf{v} \cdot \mathbf{x} = -1$ if $u, v \in V_i$ with negative edges and $\mathbf{u} \cdot \mathbf{x} = -1$, $\mathbf{v} \cdot \mathbf{x} = 1$ if $u, v \in V_i$ with positive edges. Thus, \mathbf{x} is an eigenvector of the $A_{\dot{G}}$ associated to the eigenvalue 0, 1 or -1 (depending on which cell selected vertices belong to). We record this in the following lemma.

Lemma 3.1. For $\dot{G} \in \mathcal{T}$, let $\{1, 2, \dots, h\} = I \sqcup J$, such that for all $i \in I$, V_i has negative edges, and for all $j \in J$ either the edges of V_j are positive or $s_j = 1$ (for short, V_j has no negative edges). Then:

- (i) 0 is an eigenvalue of \dot{G} with multiplicity at least $\sum_{i=1}^h (t_i - 1)$;
- (ii) 1 is an eigenvalue of \dot{G} with multiplicity at least $\sum_{i \in I} (s_i - 1)$;
- (iii) -1 is an eigenvalue of \dot{G} with multiplicity at least $\sum_{j \in J} (s_j - 1)$.

To consider the remaining eigenvalues, we first observe that the partition $U_1 \cup V_1 \cup U_2 \cup V_2 \cup \dots \cup U_h \cup V_h$ of the vertex set of \dot{G} induces an equitable partition of its adjacency matrix. The corresponding quotient matrix $Q (= Q(\dot{G}))$ is given by

$$Q(\dot{G}) = \begin{bmatrix} 0 & s_1 & 0 & s_2 & 0 & \cdots & s_{h-1} & 0 & s_h \\ t_1 & \varepsilon_1(s_1 - 1) & 0 & s_2 & 0 & \cdots & s_{h-1} & 0 & s_h \\ 0 & 0 & 0 & s_2 & 0 & \cdots & s_{h-1} & 0 & s_h \\ t_1 & s_1 & t_2 & \varepsilon_2(s_2 - 1) & 0 & \cdots & s_{h-1} & 0 & s_h \\ \vdots & & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & s_h \\ t_1 & s_1 & t_2 & s_2 & t_3 & \cdots & s_{h-1} & t_h & \varepsilon_h(s_h - 1) \end{bmatrix},$$

where $\varepsilon_k = 1$ if $k \in J$ and $\varepsilon_k = -1$ if $k \in I$. In other words,

$$Q(\dot{G})_{2i-1, 2j-1} = 0, \quad Q(\dot{G})_{2i-1, 2j} = \begin{cases} s_j & \text{if } i \leq j, \\ 0 & \text{if } i > j, \end{cases} \quad Q(\dot{G})_{2i, 2j-1} = \begin{cases} t_i & \text{if } i \geq j, \\ 0 & \text{if } i < j \end{cases}$$

and

$$Q(\dot{G})_{2i, 2j} = \begin{cases} s_i - 1 & \text{if } i = j \text{ and the edges of } V_i \text{ are positive,} \\ 1 - s_i & \text{if } i = j \text{ and the edges of } V_i \text{ are negative,} \\ s_j & \text{if } i \neq j. \end{cases}$$

We can now define two types of eigenvalues of $A_{\dot{G}}$ according to their associated eigenvectors. According to Theorem 2.1 each one of $2h$ eigenvalues of $Q(\dot{G})$ is also an eigenvalue of $A_{\dot{G}}$. These eigenvalues are of the first type: every eigenvector of $A_{\dot{G}}$ corresponding to an eigenvalue of its quotient matrix $Q(\dot{G})$ is constant on the parts of the partition, i.e. the coordinates corresponding to vertices belonging to the same cell are equal. With the procedure explained at the beginning of Section 2 we can define the $n - 2h$ linearly independent eigenvectors associated to the eigenvalues 0, 1 or -1 of the second type, all of them being orthogonal to the eigenvectors corresponding to the eigenvalues of the first type (i.e. their coordinates sum up to zero on each part of the partition). From these observations and from Lemma 3.1 and Theorem 2.1 it now easily follows that the characteristic polynomial of \dot{G} is given by

$$\phi(A_{\dot{G}}, x) = x^{t-h}(x + 1)^\alpha(x - 1)^\beta \phi(Q(\dot{G}), x), \tag{3.1}$$

$x(x + 2s_i - 1) < x(x + 1) = 0 < c_h t_i s_{i-1}$, for all $2 \leq i \leq h$. Thus, according to Theorem 2.2, we have $\det(T_{-1}) > 0$, which means that -1 is not a zero of $\phi(Q(\dot{G}), x)$, and -1 is an eigenvalue of \dot{G} with multiplicity exactly α .

Gathering the above results, we arrive at the following.

Theorem 4.1. *A signed graph $\dot{G} \in \mathcal{T}$ has no eigenvalues in $(-1, 0)$. If for any $i \in \{2, \dots, h\}$, V_i has no negative edges, then:*

- (i) *0 is an eigenvalue of \dot{G} with multiplicity $\sum_{i=1}^h (t_i - 1)$;*
- (ii) *if either $t_1 > 1$ or $t_1 = 1$ and V_1 has negative edges, then -1 is an eigenvalue of \dot{G} with multiplicity $\sum_{i \in I} (s_i - 1)$;*
- (iii) *if $t_1 = 1$ and V_1 has no negative edges, then -1 is an eigenvalue of \dot{G} with multiplicity $1 + \sum_{i \in I} (s_i - 1)$.*

Remark 4.2. If the signed threshold graph \dot{G} has no negative edges, clearly $\phi(A_{\dot{G}}, x) = \phi(A_G, x)$. In this case, Theorem 4.1 recovers the corresponding results for unsigned graphs obtained by different methods in [6, 8].

5 Eigenvalue free intervals

We determine eigenvalue free intervals under the additional assumption that $t_1 > 1$ or $t_1 = 1$ and V_1 has no negative edges. First, we consider the interval $(-\infty, 0)$.

Suppose that $t_1 = 1$ and that V_1 has no negative edges. Recall that, in this case, there must be $\det(T_{-1}) = 0$, and recall the submatrices A and B of T_x given by (4.2) and (4.3). We have $\det(T_x) = (1+x)(\det(A) - x \det(B))$, so if $x < -1$ and if $\det(A)$ and $\det(B)$ are positive, then it follows that $\det(T_x)$ is negative.

Consider now the positivity of $\det(A)$ and $\det(B)$. If the parameter c_h is defined by (4.1) and $I \sqcup J$ represents the partition of $\{1, 2, \dots, h\}$ described in Lemma 3.1, then by Theorem 2.2, we conclude that both determinants are positive whenever the conditions (2)–(5) described in Section 4 hold.

Denote further

$$a_h = \min_{2 \leq i \leq h-1} \{s_1 t_2, s_i t_{i+1}, s_i t_i, s_h t_h\}. \tag{5.1}$$

Since for $x < 0$, we have $x(x + 2s_i - 1) < x(x + 1)$, it follows that both determinants are positive whenever $x \in (\frac{-1 - \sqrt{1 + 4c_h a_h}}{2}, -1)$, which implies $\det(T_x) \neq 0$. In other words, \dot{G} has no eigenvalues in that interval.

Similarly, for $t_1 > 1$, we immediately get $\det(T_{-1}) > 0$, and by Theorem 2.2, $\det(T_x) > 0$ holds if the conditions (1)–(6) described in Section 4 hold.

Now, for $x < 0$, we have $x(x + 2s_i - 1) < x(x + 1)$, and so $\det(T_x) > 0$ whenever $x \in I_1 \cap I_2$, where

$$I_1 = \left(\frac{-1 - \sqrt{1 + 4c_h a_h}}{2}, -1 \right), I_2 = (-t_1, 0).$$

These considerations lead to the following theorem.

Theorem 5.1. *For a signed graph $\dot{G} \in \mathcal{T}$, we have:*

- (i) *if $t_1 = 1$ and V_1 has no negative edges, then \dot{G} has no eigenvalues in $(\frac{-1 - \sqrt{1 + 4c_h a_h}}{2}, -1)$;*
- (ii) *if $t_1 > 1$, then \dot{G} has no eigenvalues in $(-S, -1)$,*

where the parameters c_h and a_h are defined by (4.1) and (5.1), respectively, and $S = \min \left\{ \frac{1 + \sqrt{1 + 4c_h a_h}}{2}, t_1 \right\}$.

Since $4c_h a_h > 1$ and $\frac{-1 - \sqrt{2}}{2} > -t_1$ (for $t_1 > 1$), we have a direct consequence.

Corollary 5.2. *A signed graph $\dot{G} \in \mathcal{T}$ has no eigenvalues in $(\frac{-1 - \sqrt{2}}{2}, -1)$.*

We proceed with the assumption that $x > 0$. Here, $t_1 + x > 0$ and T_x has a positive diagonal. By Theorem 2.2, we have $\det(T_x) > 0$ whenever the conditions (1)–(5) described in Section 4 hold.

Observe that, for $x > 0$, $x(x + 1) < x(x + 2s_i - 1)$ holds, and denote

$$\mathcal{M} = \min_{2 \leq i \leq h-1} \left\{ \frac{s_1 c_h - 2s_1 + 1 + \sqrt{(2s_1 - 1 - s_1 c_h)^2 + 4c_h t_1 s_1}}{2}, \right. \\ \left. \frac{1 - 2s_i + \sqrt{(2s_i - 1)^2 + 4c_h t_i s_i}}{2}, \frac{1 - 2s_i + \sqrt{(2s_i - 1)^2 + 4c_h t_{i+1} s_i}}{2}, \right. \\ \left. \frac{1 - 2s_h + \sqrt{(2s_h - 1)^2 + 4c_h t_h s_h}}{2} \right\}. \tag{5.2}$$

It follows that $\det(T_x) > 0$ holds whenever $x \in (0, \mathcal{M})$, which leads to the following theorem.

Theorem 5.3. *A signed graph $\hat{G} \in \mathcal{T}$ has no eigenvalues in $(0, \mathcal{M})$, where the parameter \mathcal{M} is defined in (5.2).*

A similar reasoning leads to the conclusion that if G is a threshold graph, then $\det(T_x) > 0$ holds whenever the following inequalities hold: $x(x + 1) < c_h(t_1 + x)s_1$, $x(x + 1) < c_h t_j s_j$ and $x(x + 1) < c_h t_{j+1} s_j$. If $x > 0$, then we have $c_h t_1 s_1 < c_h(t_1 + x)s_1$, which implies the following.

Theorem 5.4. *A threshold graph G has no eigenvalues in $(0, \frac{-1 + \sqrt{1 + 4c_h a_h}}{2})$, where the parameters c_h and a_h are defined by (4.1) and (5.1), respectively.*

Again, the inequality $4c_h a_h > 1$ gives an immediate consequence.

Corollary 5.5. *A threshold graph G has no eigenvalues in $(0, \frac{-1 + \sqrt{2}}{2})$.*

Remark 5.6. Corollary 5.2 covers a particular case of threshold graphs. In conjunction with Corollary 5.5, it leads to the conclusion that, apart from the eigenvalues -1 and 0 , the interval $(\frac{-1 - \sqrt{2}}{2}, \frac{-1 + \sqrt{2}}{2})$ does not contain an eigenvalue of any threshold graph – a result obtained in [6].

6 Some remarks on multiplicity of eigenvalues

We conclude with the following observation. It is known from [8] that, in a threshold graph, all the eigenvalues distinct from 0 and -1 are simple. This is not true for signed threshold graphs. An example is a signed threshold graph of \mathcal{T} determined by the parameters $h = 3$, $s_1 = 2$, $s_2 = 8$, $s_3 = 1$, $t_1 = 3$, $t_2 = 1$, $t_3 = 9$, and in which, apart from those of V_1 , all the edges are positive. Its eigenvalues (distinct from 0 and -1) are: 12.9050 , 2.3678 , 0.4248 , -3.6977 and -3 with multiplicity 2 .

On the other hand, we can easily see that a signed graph of \mathcal{T} cannot have an eigenvalue (distinct from 0 and -1) with multiplicity greater than 2 . Indeed, considering the eigenvalue $\lambda \notin \{-1, 0\}$, we get $\det(T_\lambda) = 0$. Now, if we delete the last two columns and the first and the last row from (the triangular matrix) T_λ (see the end of Section 3), then we obtain a diagonal matrix with non-zero diagonal elements, so the rank of T_λ is at least $2h - 2$. Consequently, the multiplicity of λ is at most 2 .

Acknowledgement The authors would like to thank the anonymous referees for careful reading and helpful comments and suggestions that lead to the improvement of the original manuscript.

The work of the second author is supported by Serbian Ministry of Education, Science and Technological Development, Project 174033. The work of the third author is supported by Serbian Ministry of Education, Science and Technological Development, Projects 174012 and 174033.

References

- [1] C.O. Aguilar, J. Lee, E. Piato, B.J. Schweitzer, Spectral characterizations of anti-regular graphs, *Linear Algebra Appl.*, 557 (2018), 84-104.
- [2] A. Alazemi, M. Anđelić, T. Koledin, Z. K. Stanić, Eigenvalue-free intervals of distance matrices of threshold and chain graphs, *Linear Multilinear Algebra*, submitted.
- [3] A.E. Brouwer, W.H. Haemers, *Spectra of graphs*, Springer, 2011.
- [4] M. Anđelić, C. M. da Fonseca, Sufficient conditions for positive definiteness of tridiagonal matrices revisited, *Positivity*, 15 (2011), 155–159.
- [5] D. Cvetković, P. Rowlinson, S.K. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, Cambridge, 2010.
- [6] E. Ghorbani, Eigenvalue-free interval for threshold graphs, *Linear Algebra Appl.*, 583 (2019), 300-305.
- [7] D.P. Jacobs, V. Trevisan, and F. Tura, Eigenvalue location in threshold graphs, *Linear Algebra Appl.*, 439 (2013), 2762-2773.
- [8] D.P. Jacobs, V. Trevisan, F. Tura, Eigenvalues and energy in threshold graphs, *Linear Algebra Appl.*, 465 (2015), 412–425.
- [9] C.R. Johnson, M. Neumann, M.J. Tsatsomeros, Conditions for the positivity of determinants, *Linear Multilinear Algebra*, 40 (1996), 241–248.
- [10] V.R. Mahadev, U.N. Peled, *Threshold Graphs and Related Topics*, North-Holland, Amsterdam, 1995.
- [11] Z. Stanić, *Inequalities for Graph Eigenvalues*, Cambridge University Press, Cambridge, 2015.